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IMPULSE-DIFFERENTIAL ENCOUNTER GAME

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An optimal program distribution of the finite number of impulse feed-in instants for one of the players is constructed in an isotropic impulse-differential game. Sufficient uniqueness conditions for this sequence are given. The game examined in the paper can also be treated as the problem of optimal multi-impulse correction of motion. The paper continues the research in [1 — 3] and in subject matter is similar to [4 — 6].

1. Statement of the problem. Let the motions of two controlled objects (players) X and Y on a fixed time interval $[t_0, T]$ be prescribed by the differential equations with initial conditions

$$X: x^{\cdot} = \varphi(t) u, \quad x(t_0) = x^0 \tag{1.1}$$
$$Y: y^{\cdot} = \psi(t) v, \quad y(t_0) = y^0$$

Here x and y are the phase vectors of players X and Y, respectively, and u and v are their control vectors. The dimensions of vectors x, y, u and v are the same and are arbitrary. The scalar functions $\varphi(t)$ and $\psi(t)$ are prescribed, continous, nonnegative on the motion interval $[t_0, T]$ and are not identically zero. We assume that player X controls his own motion only at discrete instants $t_k, k = 1, \ldots, n,$ by feeding in impulses with limited total resource, while player Y controls his own motion on the whole interval $[t_0, T]$. The following constraints are imposed on the realizations of the controls of players X and Y and on the instants t_k :

$$u(t) = \sum_{k=1}^{n} u_k \delta(t - t_k), \quad \sum_{k=1}^{n} |u_k| \leqslant Q \tag{1.2}$$

$$t_0 \leqslant t_1 < \ldots < t_n \leqslant t_{n+1} = T \tag{1.3}$$

$$|v(t)| \leqslant 1, \quad t \in [t_0, T] \tag{1.4}$$

Here $\delta(t)$ is the delta-function, $Q > 0$ is the total resource of player X's impulse control. The strict inequalities in (1.3) do not restrict the generality since the feeding in at certain instants t_k of $l + 1$ successive impulses of intensities $u_k, \ldots, u_{k+l}$ is equivalent to the feeding in of one impulse of intensity $u_k + \ldots + u_{k+l}$. Thus, player X's phase vector undergoes jumps of magnitude $\varphi(t_k) u_k$ at instants t_k, $k = 1, \ldots, n,$ that are assumed fixed for the time being. Player X strives to minimize the distance between the players at the final instance T, i.e., the functional

$$J = |x(T) - y(T)| \tag{1.5}$$

Player Y obstructs this by realizing integrable controls $v(t)$ subject to constraints

(1.4); such controls are said to be admissible and are denoted by v for brevity.

We introduce the notation

$$x_k = x(t_k - 0),\ y_k = y(t_k - 0), \quad k = 2,\ldots,n \tag{1.6}$$
$$x_1 = x(t_1 - 0),\ y_1 = y(t_1 - 0), \quad t_1 > t_0$$
$$x_1 = x(t_0) = x^\circ,\ y_1 = y(t_0) = y^\circ, \quad t_1 = t_0$$

By q_k we denote player X's control resource available before the k-th impulse

$$q_1 = Q, \quad q_k = Q - \sum_{i=1}^{k-1} |u_i|, \quad k = 2, \ldots, n \tag{1.7}$$

The totality of quantities (x_k, y_k, q_k) that completely characterized the state of objects (1.1) immediately before the feeding in of the k-th impulse at the instant $t_k - 0$ is called a position.

We assume that before each impulse is fed in player X observes the position realized and chooses the jump vectors in the form of functions $u_k = u_k(x_k, y_k, q_k)$, $k = 1, \ldots, n$, i.e., applies a position control. Since q_k is the amount of available resource, the functions indicated must satisfy the constraint

$$|u_k(x, y, q)| \leqslant q, \quad k = 1, \ldots, n \tag{1.8}$$

for any x and y. It is obvious that it suffices to determine the third argument q of the function $u_k(x, y, q)$ within the range $0 \leqslant q \leqslant Q$. The aggregate of functions $u_k(x, y, q)$, $k = 1, \ldots, n$, satisfying constraint (1.8) is called an admissible strategy of player X and for brevity is denoted by u. To each pair (u, v) consisting of an admissible strategy u of player X and an admissible control v of player Y corresponds a unique solution of Eqs. (1.1) and a value $J[u, v]$ of functional (1.5).

Problem 1. Find the optimal guaranteeing strategy u^* of player X and the minimum guaranteed value J^* of functional (1.5), satisfying the relation

$$J^* = \min_u \sup_v J[u, v] = \sup_v J[u^*, v] \tag{1.9}$$

The minimum here is computed over all admissible strategies and the upper bound is computed over all admissible controls.

Using the variable $z(t) = x(t) - y(t)$, the equation of motion with initial conditions (1.1) and the functional (1.5) are rewritten as

$$z^\cdot = \varphi(t)\, u - \psi(t)\, v,\ z(t_0) = z^\circ = x^\circ - y^\circ;\ J = |z(T)| \tag{1.10}$$

The vector v in (1.10) can be treated as an unknown perturbance subject to constraint (1.4) and the vector u can be treated as a correcting impulse control of form (1.2), (1.3). Then the game Problem 1 is also a problem on the optimal minimax correction of the motion of (1.10), having the purpose of minimizing the final miss $z(T)$ ([1,2]). With the aid of (1.10) we can perceive that the optimal strategy u^* belongs to the class of strategies of the form $u_k(z, q)$, $z = x - y$, $k = 1, \ldots, n$.

2. **Equivalent multistep game.** We introduce the notation

$$x_{n+1} = x(T+0), \quad y_{n+1} = y(T+0), \quad \rho(t) = \int_t^T \psi(\tau)\, d\tau \qquad (2.1)$$
$$z_k = x_k - y_k, \quad k = 0, \ldots, n+1$$

We assume in addition that the inequality $\psi(t) > 0$ holds on some interval $(T - \varepsilon, T)$, $\varepsilon > 0$. It then follows from (2.1) that $\rho(t) > 0$ for $t_0 \leqslant t < T$, $\rho(T) = 0$. Only the variables z_k and q_k are used for constructing the optimal strategy in Problem 1 and the game's result. Therefore, we can restrict our attention to the following multistep game (see [3]) with phase variables z_k and q_k and controls u_k (of player X) and v_k (of player Y)

$$z_{k+1} = z_k + \varphi_k u_k - (\rho_k - \rho_{k+1}) v_k, \quad q_{k+1} = q_k - |u_k| \qquad (2.2)$$
$$J = |z_{n+1}|, \quad z_0 = z^\circ, \quad q_0 = Q, \quad u_0 = 0$$
$$|u_k| \leqslant q_k, \quad |v_k| \leqslant 1, \quad \varphi_k = \varphi(t_k), \quad \rho_k = \rho(t_k), \quad k = 0, \ldots, n$$

The dynamic equations are obtained by integrating relations (1.10) and using equalities (1.7). The equality $u_0 = 0$ in (2.2) reflects the fact that up to the instant t_1 player X does not control his own motion. Player X's strategies in game (2.2) are analogous to those described above; every sequence of vectors v_k, $|v_k| \leqslant 1$, $k = 0, 1, \ldots, n$, serves as a control of player Y. Game (2.2) is equivalent to the original impulse-differential game (and to the correction game (1.10)) in the following sense. The equalities

$$v_k = [\rho_k - \rho_{k+1}]^{-1} \int_{t_k}^{t_{k+1}} \psi(\tau) v(\tau)\, d\tau, \quad k = 0, 1, \ldots, n \qquad (2.3)$$
$$v(t) = v_k, \quad t \in (t_k, t_{k+1}], \quad v(t_0) = v_0$$

establish a one-to-one correspondence between the controls $v(t)$ in the original game and the controls v_k in game (2.2), such that one and the same sequences z_k, $k = 0, \ldots, n+1$ and, consequently, equal values of the functional, are realized in both games for every strategy of player X.

We introduce into consideration the Bellman function $S_k(z, q)$, $k = 0, \ldots, n+1$, equal to the minimum guaranteed value of functional $|z_{n+1}|$ under the condition that the multistep game (2.2) starts at the k-th step from the point $z_k = z$ with player X's control resource reserve equalling q. In particular, $S_0(z^\circ, Q) = J^*$, where J^* is defined in (1.9). The Bellman function satisfies the following recurrence relation with boundary condition [3]:

$$S_k(z_k, q_k) = \min_{|u_k| \leqslant q_k} \max_{|v_k| \leqslant 1} S_{k+1}(z_{k+1}, q_{k+1}) \qquad (2.4)$$
$$k = 0, 1, \ldots, n, \quad S_{n+1}(z_{n+1}, q_{n+1}) = |z_{n+1}|$$

Player X's optimal strategy (the solution of problem 1) yields the minimum in (2.4). We determine the quantities φ_k^* by the equalities

$$\psi_k^* = \max_{k \leqslant i \leqslant n} \varphi_i, \quad k = 1, \ldots, n, \quad \varphi_0^* = \varphi_1^*, \quad \varphi_{n+1}^* = 0 \qquad (2.5)$$

Lemma 1. Recurrence relations (2.4) have the unique solution

$$S_k(z, q) = \max_{k \leqslant m \leqslant n+1} f_{k,m} \tag{2.6}$$

$$f_{k,m} = \varphi_m^* \left[\frac{|z|}{\varphi_k^*} - q + \sum_{i=k+1}^{m} \frac{\rho_{i-1} - \rho_i}{\varphi_i^*} \right] + \rho_m$$

$$m = k, \ldots, n, \quad f_{k,n+1} = \rho_n, \quad k = 0, \ldots, n$$

Player X 's optimal strategy is determined by the equalities

$$\begin{aligned} u_k^*(z, q) &= -z / \varphi_k, \quad |z| \leqslant \varphi_k q \\ u_k^*(z, q) &= -\frac{z}{|z|} q, \quad |z| > \varphi_k q, \quad \varphi_k > \varphi_{k+1}^* \\ u_k^*(z, q) &= 0, \quad |z| > \varphi_k q, \quad \varphi_k \leqslant \varphi_{k+1}^* \end{aligned} \tag{2.7}$$

The lemma can be proved by mathematical induction, using (2.4).

Player Y 's optimal control in (2.2) (the worst one from player X 's view-point) is found during the computation of the maximum in (2.4) and is

$$\begin{aligned} v_k^* &= -\frac{z_k + \varphi_k u_k}{|z_k + \varphi_k u_k|}, \quad z_k + \varphi_k u_k \neq 0 \\ v_k^* &= e, \quad z_k + \varphi_k u_k = 0; \quad k = 0, \ldots, n \end{aligned} \tag{2.8}$$

(e is an arbitrary unit vector).

From (1.10) we obtain

$$z(t_k + 0) = z_k + \varphi_k u_k, \quad k = 1, \ldots, n \tag{2.9}$$

We derive player Y 's optimal control in terms of the original game by using relations (2.3), (2.8) and (2.9):

$$v^*(t) = -\frac{z(t_k + 0)}{|z(t_k + 0)|}, \quad z(t_k + 0) \neq 0 \tag{2.10}$$

$$v^*(t) = 0, \; z(t_k + 0) = 0, \; t \in (t_k, t_{k+1}], \quad k = 0, 1, \ldots, n$$

3. Optimization of the impulse feed-in instants. Formula (2.6) with $k = 0$, $z = z^\circ$ and $q = Q$ determines the minimal value of functional (1.5), guaranteed to player X, for an arbitrary program (specified before the start of the game) distribution of impulse feed-in instants.

Problem 2. Find the optimal program sequence of instants t_k^*, $k = 1, \ldots, n$, such that under constraints (1.3)

$$\min_{(t_1, \ldots, t_n)} J^* = J^\circ \tag{3.1}$$

The minimum in (3.1) exsists on the closure of set (1.3) since the quantity $J^* = S_0(z^\circ, Q)$ in (2.6) is a continuous function of variables $t_1, \ldots, t_n$. If the minimum in (3.1) is reached at a boundary point of set (1.3), then, as follows from the remark in Sect. 1, we can find another minimum point satisfying conditions (1.3).

By t_* we denote the point of maximum of function $\varphi(t)$ on interval $[t_0, T]$, closest to instant T

$$\max_t \varphi(t) = \varphi(t_*), \ t_0 \leqslant t \leqslant T \tag{3.2}$$

Using formula (2.6) we can show that the magnitude of J^* for some sequence $t_1, \ldots, t_n$ does not increase if all the instants t_k lying in the interval $[t_0, t_*]$ combine with instant t_*. Consequently, the optimal sequence t_k^* is found among the sequences satisfying the condition

$$t_* \leqslant t_1 < \ldots < t_n \leqslant t_{n+1} = T \tag{3.3}$$

Then the minimum in (3.1) under constraints (1.3) coincides with the minimum under constraints (3.3).

Let us show the ranges of the parameters, for which the minimum in (3.1) is easily computed. More precisely, let the inequality

$$r^\circ - Q\varphi(t_*) + \rho(t_0) - \rho(t_*) \geqslant 0, \ r^\circ = |z^\circ| \tag{3.4}$$

be fulfilled. Relations (3.3) and (3.4) permit us to establish the estimate

$$r^\circ - Q\varphi(t_1) + \rho(t_0) - \rho(t_1) \geqslant r^\circ - Q\varphi(t_*) + \rho(t_0) - \rho(t_*) \geqslant 0$$

with whose aid we get that the quantities (2.6) satisfy the conditions $f_{0,1} \geqslant \ldots \geqslant f_{0,n+1}$ on any sequence of form (3.3). Consequently, we have $J^* = f_{0,1}$. From relations (3.1) –(3.3) we then derive

$$J^\circ = \min_{t_1} [r^\circ - Q\varphi(t_1) + \rho(t_0)] = r^\circ - Q\varphi(t_*) + \rho(t_0) \quad t_1^* = t_* \tag{3.5}$$

Thus, in case (3.4) player X 's first impulse should be fed in at instant t_*. If player Y applied the optimal control (2.9) on the interval $[t_0, t_*]$, then according to (2.7) and (3.4) player X will have used up all of resource Q on the impulse indicated. The remaining instants of feeding in the (zero) impulses can be chosen arbitrarily within the scope of constraints (3.3).

Now let the inequality

$$r^\circ - Q\varphi(t_*) + \rho(t_0) - \rho(t_*) < 0 \tag{3.6}$$

opposite to (3.4), be fulfilled. The theorem following below gives an algorithm for constructing the optimal sequence under condition (3.6). In the formulation and proof of this theorem we introduce the nonnegative functions

$$\Phi_k(t_1, \ldots, t_k) = \frac{r^\circ}{\varphi_1^*} + \sum_{i=1}^{k} \frac{\rho_{i-1} - \rho_i}{\varphi_i^*}, \quad k = 1, \ldots, n \tag{3.7}$$

and establish their properties. We examine sequences for which $\varphi(t_n) = \varphi_n^* > 0$. We note that for finding the quantities φ_k^* by formula (2.5) it is necessary to prescribe a complete sequence $t_1, \ldots, t_n$ and to assume that the quantities Φ_k are functions of only the first k terms of the sequence. The minima

$$\Phi_k^*(t_k) = \min_{(t_1,\ldots,t_{k-1})} \Phi_k(t_1,\ldots,t_k), \quad t_* \leqslant t_1 \leqslant \ldots \leqslant t_k, \quad k=2,\ldots,n \tag{3.8}$$

are reached on the sequence $t_1', \ldots, t_{k-1}'$, possibly not unique, satisfying the conditions

$$t_* \leqslant t_1' < \ldots < t_{k-1}' < t_k \tag{3.9}$$

$$\varphi_1^* > \varphi_2^* > \ldots > \varphi_k^*, \quad k=2,\ldots,n \tag{3.10}$$

Using definition (2.5), from inequalities (3.10) we can obtain $\varphi(t_1') > \ldots > \varphi(t_{k-1}') > \varphi(t_k)$. Thus, the instants t_i' are distributed in such a way that the sequence of values $\varphi(t_i')$ decreases strictly monotonically. Properties (3.9) and (3.10) of the sequence t_i' can be derived from the inequality

$$\frac{\rho_{i-1}-\rho_{i+1}}{\varphi_{i+1}} \geqslant \frac{\rho_{i-1}-\rho_i}{\varphi_i} + \frac{\rho_i-\rho_{i+1}}{\varphi_{i+1}} \tag{3.11}$$

valid for the three instants $t_{i-1} < t_i < t_{i+1}$ for which $\varphi_{i-1} > \varphi_i > \varphi_{i+1}$.

We set $\Phi_1^*(t_1) = \Phi_1(t_1)$. Using relations (3.7) and (3.8) we can show that the functions $\Phi_k^*(\tau)$ do not decrease as τ grows, i.e.,

$$\Phi_k^*(\tau_1) \leqslant \Phi_k^*(\tau_2), \quad \tau_1 < \tau_2, \; k=1,\ldots,n \tag{3.12}$$

Let $t_1^\circ, \ldots, t_{n-1}^\circ$ be a sequence yielding the minimum in (3.8) with $t_1 = t_1^\circ$ and with some $t_n = t_n^\circ$ Then, obviously, when $t_k = t_k^\circ$, $k = 2, \ldots, n-1$ minima (3.8) are reached on the sequence $t_1^\circ, \ldots, t_{k-1}^\circ$ and the inequalities

$$\Phi_1^*(t_1^\circ) \leqslant \Phi_2^*(t_2^\circ) \leqslant \ldots \leqslant \Phi_n^*(t_n^\circ) \tag{3.13}$$

hold. Using (3.7), the quantities $f_{0,k}$ from formula (2.6) can be rewritten as

$$f_{0.k} = \varphi_k^* [\Phi_k(t_1, \ldots, t_k) - Q] + \rho_k, \quad k=1,\ldots,n \tag{3.14}$$

Since

$$\min_{\substack{(t_1,\ldots,t_{k-1}) \\ t_* \leqslant t_1 \leqslant \ldots \leqslant t_{k-1} \leqslant t_k}} f_{0,k} = \varphi_k^* [\min_{(t_1,\ldots,t_{k-1})} \Phi_k(t_1,\ldots,t_k) - Q] + \rho_k \tag{3.15}$$

the minimum in (3.15) is reached on sequence (3.9) yielding the minimum in (3.8).

Theorem 1. Let inequality (3.6) be fulfilled. Then:

1°. If the inequality

$$Q < Q^\circ = \lim_{\tau\to T} \Phi_n^*(\tau) \tag{3.16}$$

holds, the optimal sequence, possibly not unique, satisfies the relations

$$\Phi_n^*(t_n^*) = Q \tag{3.17}$$

$$\min_{(t_1,\ldots,t_{n-1})} \Phi_n(t_1,\ldots,t_{n-1},t_n^*) = \Phi_n(t_1^*,\ldots,t_n^*) \tag{3.18}$$

$$t_* \leqslant t_1 < \ldots < t_{n-1} < t_n^*$$

The quantity $J^\circ = \rho(t_n{}^*)$.

2°. If the inequalities

$$\varphi(T) > 0, \quad Q \geqslant Q^\circ \tag{3.19}$$

hold, every sequence of form (3.3) satisfying the conditions

$$\Phi_n(t_1, \ldots, t_n) \leqslant Q, \quad t_n = T \tag{3.20}$$

is optimal. The optimal sequence is not unique under the strict inequality $Q > Q^\circ$. The quantity J° is:

$$J^\circ = \rho(t_n{}^*) = \rho(T) = 0 \tag{3.21}$$

Proof. Using (2.6) and (3.3), we transform relation (3.1) to

$$J^\circ = \min_{(t_1, \ldots, t_n)} \max_{1 \leqslant k \leqslant n+1} f_{0,k} = \min_{t_* \leqslant t_n \leqslant T} \max\{\lambda(t_n), \rho(t_n)\} \tag{3.22}$$

$$\lambda(t_n) = \min_{(t_1, \ldots, t_{n-1})} \max_{1 \leqslant k \leqslant n} f_{0,k}, \quad t_* \leqslant t_1 < \ldots < t_{n-1} < t_n \tag{3.23}$$

Here we have taken into account that $f_{0,n+1} = \rho(t_n)$. According to (2.1), $\rho(t_n)$ is a nonincreasing function of t_n in the interval $[t_0, T]$ and $\rho(T) = 0$. Using (3.23), (3.6) and (2.6) we find that

$$\lambda(t_*) = r^\circ - Q\varphi(t_*) + \rho(t_0) < \rho(t_*) \tag{3.24}$$

From the noted properties of the continuous functions $\lambda(t_n)$ and $\rho(t_n)$ it follows that the minimum over the t_n in (3.22) is reached either at the point $t_n = t_n{}^*$ which is the maximum (closest to T) root of the equation

$$\lambda(t_n) = \rho(t_n) \tag{3.25}$$

or at the point $t_n = t_n{}^* = T$. Inequality (3.24) implies the condition $t_n{}^* > t_*$.

We turn to the proof of statement 1°. We note that $Q^\circ = +\infty$ holds when $\varphi(T) = 0$ and that the condition $Q < Q^\circ$ is fulfilled for every finite Q. By $t_n = t_n'$ we denote the root closest to T of the equation

$$\Phi_n{}^*(t_n) = Q \tag{3.26}$$

By virtue of the monotonicity in (3.12) the root of Eq. (3.26) under conditions (3.6) and (3.16) exsists, and $t_* < t_n' < T$. It can be shown that function $\lambda(t_n)$ satisfies the conditions

$$\lambda(t_n) = \varphi(t_n)[\Phi_n{}^*(t_n) - Q] + \rho(t_n), \quad t_* \leqslant t_n \leqslant t_n' \tag{3.27}$$
$$\lambda(t_n) > \rho(t_n), \quad t_n' < t_n \leqslant T$$

From formulas (3.27) it follows that Eqs. (3.25) and (3.26) are equivalent, i.e., $t_n' = t_n{}^*$, since equality (3.25) cannot be satisfied when $\varphi(t_n) = 0$. Therefore, in particular, $\varphi(t_n{}^*) > 0$. Equalities (3.17) and (3.18) are proved. The inequality $\rho(t_n{}^*) > 0$ follows from the inequality $t_n{}^* < T$ noted above. Statement 1° has been proved.

To prove statement 2° it remains to show that under conditions (3.19) the minimum in (3.22) over the t_n is reached at the point $t_n = T$. Using relations (2.6) and (3.22), we get that the relations $f_{0,1} \leqslant \ldots \leqslant f_{0,n} \leqslant 0$, $\lambda(T) \leqslant 0$ and $J^\circ = \rho(T) = 0$ hold on any sequence $t_1, \ldots, t_n$ of form (3.3) satisfying constraints (3.20). Conditions (3.20) are satisfied, in particular, by the sequence yielding minimum (3.18) with $t_n^* = T$. For this sequence the inequalities (3.20) and $\lambda(T) \leqslant 0$ are strict when $Q > Q^\circ$. Therefore, these inequalities are not violated under a variation δt_k of instant t_k, sufficiently small in absolute value, $k = 1, \ldots, n-1$. Hence follows the nonuniqueness of the optimal sequence in case (3.19).

If during the game player Y deviates from control (2.10), then in order to take advantage of the opponent's "failures" player X, before feeding in the next impulse, must recompute the optimal program distribution of the impulse feed-in instants. Such a recomputation can in principle be carried out using the synthesis function $t_1^* = \vartheta(r, t, n, Q)$ equal to the optimal feed-in instant of the first impulse from the n impulses at hand, under the condition that the game begins at instant t from the point $z = x - y$, $|z| = r$ with player X's resource reserve equal to Q. An algorithm for using function ϑ has been presented in [1,2].

Let us formulate a sufficient condition for the uniqueness of the sequence $t_1^*, \ldots, t_n^*$, constructed in Theorem 1.

Lemma 2. Let the derivatives $\varphi^\cdot$, $\varphi^{\cdot\cdot}$ and $\psi^\cdot$ exist and be continuous in the interval (t_*, T) and let

$$\varphi^\cdot(t) < 0,\ \varphi^{\cdot\cdot}(t) \leqslant 0,\ \psi^\cdot \leqslant 0,\ t_* < t < T \tag{3.28}$$

Then the optimal sequence $t_1^*, \ldots, t_n^*$ is unique when $Q \leqslant Q^\circ$.

Using formulas (2.7) and (2.10) it can be shown that under the hypotheses of Lemma 2 all impulses of player X when player Y uses his optimal control are nonzero and are determined by equalities (2.7) for the case $|z| \leqslant \varphi_k q$. The quantity $\Phi_n^*(t_n^*)$ in Eq. (3.17) equals the sum $|u_1^*| + \ldots + |u_n^*|$. From (2.9) and (2.7) it follows that the equality $z(t_k^* + 0) = 0$ holds after each impulse, i.e., the optimal impulses compensate for the deviations from zero of vector $z(t)$. Relations (2.7) and (3.17) enable us to investigate player X's optimal strategy in the limiting case as $n \to \infty$ analogously as shown in [1].

Let us consider game (1.1) – (1.5) from the point of view of player Y, i.e., the maximin problem for functional (1.5). Using (2.10), we offer the following position strategy of player Y:

$$v(t) = -z(t)/|z(t)|,\ z(t) \neq 0;\ v(t) = e,\ z(t) = 0 \tag{3.29}$$

It can be shown that this strategy guarantees player Y a value of functional (1.5) not less than the J^* in (1.9) at the fixed instants t_k and not less than the J° in (3.1) at the instants not fixed. Thus, a saddle situation, defined by strategies (2.7) and (3.29), exists in the position game (1.1) – (1.5); and in the game with nonfixed instants t_k, player X constructs them by using the synthesis function ϑ. In other words, the minimax in (1.9) is permutational even if the minimization is carried out also over the instants t_k (see (3.1)).

We remark that, for example, the dynamic equations in the encounter game for objects X and Y, defined by the differential equations $L^m(t)\, x = u$ and

$L^k(t)\,y = v$, reduce to equations of motion of form (1.1) or (1.10). Here $L^m(t)$ and $L^k(t)$ are linear scalar differential operators of order m and k. Concrete systems of such kind have been investigated in [1 – 3] for $m = k = 2$ ([1, 2]) and for $m = 2$ and $k = 1$ ([1,3]).

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